

The implicitization problem for $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$

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Abstract

We develop in this paper methods for studying the implicitization problem for a rational map $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$ defining a hypersurface in $(\mathbb{P}^1)^{n+1}$, based on computing the determinant of a graded strand of a Koszul complex. We show that the classical study of Macaulay Resultants and Koszul complexes coincides, in this case, with the approach of approximation complexes and we study and give a geometric interpretation for the acyclicity conditions.

Under suitable hypotheses, these techniques enable us to obtain the implicit equation, up to a power, and up to some extra factor. We give algebraic and geometric conditions for determining when the computed equation defines the scheme theoretic image of ϕ , and, what are the extra varieties that appear. We also give some applications to the problem of computing sparse discriminants.

Key words: elimination theory, rational map, syzygy, approximation complex, Koszul complex, implicitization.

1. Introduction

In this work we study the implicitization problem for a finite rational map $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$ over a field k , hence, its image is a hypersurface in $(\mathbb{P}^1)^{n+1}$. Having a rational map $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$ is equivalent to having $n + 1$ pairs of homogeneous polynomials f_i, g_i of the same degree d_i , for $i = 0, \dots, n$, f_i, g_i with no common factors.

We show that the classical study of Macaulay Resultants and Koszul complexes coincides with the new approach introduced by L. Busé and J.-P. Jouanolou in (BJ03) and developed by them and M. Chardin in (Cha06; BC05; Bus06; BCJ06), by means of approximation complexes, defined by J. Herzog, A. Simis and W. Vasconcelos in (HSV82) and (HSV83).

The process consists of computing the implicit equation by means of the classical methods of elimination theory adapted for this case. More precisely, we will consider the multigraded k -algebra \mathcal{B} that corresponds to the incidence variety associated to the given rational map ϕ . This algebra can be presented as a quotient of the polynomial ring R in all the groups of variables, by some linear equations L_0, \dots, L_n . Consequently

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we propose as a resolution for \mathcal{B} , the Koszul complex $\mathcal{K}_\bullet^R(L_0, \dots, L_n)$, denoted by \mathcal{K}_\bullet , and we study and give a geometric interpretation of its acyclicity conditions.

In this case, we obtain the implicit equation (up to a power) by taking the determinant of a suitable strand of a multigraded resolution, that is:

$$H^{\deg(\phi)} = \text{Res}(L_0, \dots, L_n) = \det((\mathcal{K}_\bullet)_\nu), \quad \text{for } \nu \gg 0.$$

Later, we analyze the geometrical meaning of these results. We give algebraic and geometric conditions for knowing when, the computed equation defines the scheme theoretic image of ϕ . And, when it is not, we present a careful analysis of the extra varieties that appear.

Finally, we give some applications to the problem of computing sparse discriminants, or A -discriminants (cf. (CD07)), by means of implicitization techniques, that were one of the reasons for developing this technique.

The key point of our approach is that the hypotheses in our main Theorem 13 are generically satisfied for rational parametrizations whose coordinates are rational functions of degree zero (defining naturally a rational morphism to $(\mathbb{P}^1)^{n+1}$), while if we try to reduce this situation to the standard case of a rational morphism to \mathbb{P}^n by taking a common denominator, bad base points appear in general and the results developed in (BJ03) do not apply.

2. Preliminaries on commutative algebra

2.1. The Koszul complex

We present here some basic tools of commutative algebra we will need for our purpose, starting by a classical result due to Hurwitz (1913). He showed that in the generic case, when there are at least as many variables as homogeneous polynomials, the Koszul complex is acyclic.

Let R be a ring, and P_0, \dots, P_n a sequence in R generating an ideal that we will denote by I . Denote by \mathcal{K}_\bullet the associated Koszul complex $\mathcal{K}_\bullet(P_0, \dots, P_n; R)$:

$$\mathcal{K}_\bullet : 0 \rightarrow \bigwedge^{n+1} R^{n+1} \xrightarrow{\partial_{n+1}} \dots \rightarrow \bigwedge^{i+1} R^{n+1} \xrightarrow{\partial_{i+1}} \bigwedge^i R^{n+1} \rightarrow \dots \xrightarrow{\partial_1} \bigwedge^0 R^{n+1} \rightarrow 0,$$

where the morphisms $\partial_{i+1} : \bigwedge^{i+1} R^{n+1} \rightarrow \bigwedge^i R^{n+1}$, are defined in such way that the element $e_{k_1} \wedge \dots \wedge e_{k_{i+1}} \in \bigwedge^{i+1} R^{n+1}$ is mapped to $\sum_{j=1}^{i+1} (-1)^{j-1} P_{k_j} e_{k_1} \wedge \dots \wedge \widehat{e_{k_j}} \wedge \dots \wedge e_{k_{i+1}}$.

If R is graded and every P_i is homogeneous of degree $d_i > 0$, this complex inherits the grading. If we introduce in \mathcal{K}_\bullet the grading given by $\deg(e_{k_1} \wedge \dots \wedge e_{k_{i+1}}) = d_{k_1} + \dots + d_{k_{i+1}}$, the differentials are of degree 0. It is a well known fact that, P_0, \dots, P_n is a regular sequence of homogeneous polynomials of positive degree if and only if \mathcal{K}_\bullet is acyclic (see for instance (Bou07, Sec. 9, N.7, Cor. 2)). Naturally, if R is graded or multigraded, and the P_i are homogeneous with respect to the grading or multigrading, this complex inherits this multigrading, and the same statement of acyclicity still holds. We will apply this to the particular case where $R = k[t_0, \dots, t_n] \otimes_k k[X_0, Y_0, \dots, X_n, Y_n]$, and k any field. Here, the ring is naturally bigraded, and can also be seen as $\mathbb{N} \times \mathbb{N}^{n+1}$ -graded, by considering one grading given by t_0, \dots, t_n , and the \mathbb{N}^{n+1} -grading given by the $n+1$ pairs X_i, Y_i . The polynomials P_i will be hence multihomogeneous of multidegree $(d_i, 0, \dots, 1, \dots, 0)$, precisely $P_i \in k[t_0, \dots, t_n]_{d_i} \otimes_k k[X_i, Y_i]_1$, and will be called L_i because of their linearity in the second group of variables.

Assume A is a noetherian commutative ring, and $R = A[X_1, \dots, X_m]$. Set $I = (P_0, \dots, P_n)$, with $P_i = \sum_{j=1}^m m_{ij} X_j$, $m_{ij} \in A$. A theorem due to L. Avramov (Avr81) gives necessary and sufficient conditions for I to be a complete intersection in R in terms of the depth of certain ideals of minors of $M := (m_{ij})_{i,j} \in \text{Mat}_{m,n+1}(A)$.

Theorem 1. [L. Avramov] *The ideal I is a complete intersection in R if and only if for all $r \in \{0, \dots, n\}$, $\text{codim}_A(I_r) \geq (n+1) - r + 1$. Here $I_r = I_r(M)$ is the ideal of A generated by the $r \times r$ minors of M , for $0 \leq r \leq r_0 := \min\{n+1, m\}$. We define $I_0 := A$ and $I_r = 0$ for $r > r_0$.*

For a proof we refer the reader to (Avr81, Prop. 1).

Later in this article, we will apply this result when $A = k[t_0, \dots, t_n]$, $R = A[X_0, Y_0, \dots, X_n, Y_n]$, and M is the $2(n+1) \times (n+1)$ matrix

$$M = \begin{pmatrix} -g_0 & \dots & 0 \\ f_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -g_n \\ 0 & \dots & f_n \end{pmatrix}, \quad (1)$$

that defines a map of A -modules $\psi : A^{n+1} \rightarrow \bigoplus_{i=0}^n A[x_i, y_i]_1 \cong A^{2(n+1)}$, and we see that the symmetric algebra $\text{Sym}_A(\text{coker}(\psi)) \cong A[\mathbf{X}]/(P_0, \dots, P_n)$, where \mathbf{X} stands for the variables $x_0, y_0, \dots, x_n, y_n$. As A is a graded k -algebra, $\text{Sym}_A(\text{coker}(\psi))$ is naturally multigraded and the graph of ϕ is an irreducible component of $\text{MultiProj}(\text{Sym}_A(\text{coker}(\psi))) \subset \mathbb{P}^n \times (\mathbb{P}^1)^{n+1}$.

2.2. Approximation complexes

We give a brief outline of the construction of the approximation complexes of cycles, and show that the complex \mathcal{Z}_\bullet coincides in this particular case (under weak hypotheses) with a certain Koszul complex.

Assume we are given a sequence $\mathbf{f} := (f_0, \dots, f_{n+1})$ of homogeneous elements of degree d over the graded ring $A = k[t_0, \dots, t_n]$, generating an ideal I . Consider the two Koszul complexes $\mathcal{K}_\bullet^A(\mathbf{f})$ and $\mathcal{K}_\bullet^R(\mathbf{f}) = \mathcal{K}_\bullet^A(\mathbf{f}) \otimes_A R$. We have also, in $A[x_0, \dots, x_{n+1}]$, another relevant sequence to consider, let's call it $\mathbf{X} := (x_0, \dots, x_{n+1})$, and we can consider also the corresponding Koszul complex $\mathcal{K}_\bullet^R(\mathbf{X})$, which is of course acyclic because of the regularity of the sequence \mathbf{X} .

A straightforward computation permits to verify that their differential anticommutes, i.e. $d_f \circ d_X + d_X \circ d_f = 0$, and this implies that d_X induces a differential on the cycles, boundaries and homologies of the complex $\mathcal{K}_\bullet^R(\mathbf{f})$. We obtain in this way three complexes noted by \mathcal{Z}_\bullet , \mathcal{B}_\bullet and \mathcal{M}_\bullet , called respectively the approximation complexes of cycles, boundaries and homologies.

Remark 2. Recall that the homology modules of these complexes are, up to isomorphism, independent of the choice of generators for I . (See for instance (Vas94, Cor 3.2.7)).

A more explicit description of the \mathcal{Z} -complex is the following:

$$\mathcal{Z}_\bullet(\mathbf{f}) : 0 \rightarrow Z_{n+1}(\mathcal{K}_\bullet^R(\mathbf{f}))[d(n+1)](-n-1) \xrightarrow{d_X} \dots \xrightarrow{d_X} Z_1(\mathcal{K}_\bullet^R(\mathbf{f}))[d](-1) \xrightarrow{d_X} A[\mathbf{X}] \rightarrow 0$$

where $Z_i(\mathcal{K}_\bullet^R(\mathbf{f}))$ stands for the i th cycle of the complex $\mathcal{K}_\bullet^R(\mathbf{f})$, and $[-]$ and $(-)$ are the shifts in the two groups of variables t_0, \dots, t_n and x_0, \dots, x_{n+1} .

This complex has as objects the bigraded modules

$$\mathcal{Z}_i = Z_i(\mathcal{K}_\bullet(\mathbf{f}; A))[di] \otimes_A R(-i), \quad (2)$$

and in the future let Z_i denote the module $Z_i(\mathcal{K}_\bullet(\mathbf{f}; A))$.

In the case of a two-generated ideal, one has the following:

Proposition 3. *With the notation above, if the sequence $\{f, g\}$ is regular, then there exists a bigraded isomorphism of complexes*

$$\mathcal{Z}_\bullet(f, g) \cong \mathcal{K}_\bullet(L; A[x, y]),$$

where $L := yf - xg$.

Proof. Given the sequence $\{f, g\}$ the approximation complex is:

$$\mathcal{Z}_\bullet(f_0, g_0) : 0 \rightarrow Z_1[d] \otimes_A A[x, y](-1) \xrightarrow{(x, y)} Z_0 \otimes_A A[x, y] \rightarrow 0.$$

As the sequence $\{f, g\}$ is regular, $H_1(\mathcal{K}_\bullet^A(\mathbf{f})) = 0$, hence $Z_1 = (-g, f)A \cong A$ by the isomorphism ψ that maps $a \in A$ to $(-g.a, f.a) \in Z_1$, given by the left morphism of the acyclic Koszul complex $\mathcal{K}_\bullet^A(\mathbf{f})$. Tensoring with $A[x, y]$ we get an isomorphism of A -modules, $\mathcal{Z}_1 \cong A[x, y]$.

The commutativity of the following diagram shows that $\mathcal{Z}_\bullet(f, g) \cong \mathcal{K}_\bullet(L; A[x, y])$

$$\begin{array}{ccccccc} \mathcal{Z}_\bullet : & 0 & \longrightarrow & Z_1[d] \otimes_A A[x, y](-1) & \xrightarrow{(x, y)} & Z_0[d] \otimes_A A[x, y] & \longrightarrow 0 \\ & & & \uparrow \scriptstyle \psi \otimes_A 1_{A[x, y]} & & \uparrow \scriptstyle = & \\ \mathcal{K}_\bullet : & 0 & \longrightarrow & A[x, y](-d) & \xrightarrow{L} & A[x, y] & \longrightarrow 0, \end{array}$$

where \mathcal{K}_\bullet denotes $\mathcal{K}_\bullet(L; A[x, y])$, $[-]$ denotes the degree shift for the grading on A and $(-)$ the shift in the variables x, y . \square

2.3. Elimination theory, the Macaulay Resultant and the U-Resultant

We recall here some basic properties of elimination theory, and also basic facts about resultants that were introduced by F. S. Macaulay in 1902, and later formalized by J.-P. Jouanolou in “Le formalisme du Résultant”, cf. (Jou91). This resultant corresponds to a generalization of the Sylvester Resultant of two homogeneous polynomials in two variables. Here, we present a brief outline in elimination theory and its classical results.

Let \mathbb{Z} be the ring of integers, $\mathbf{t} = t_0, \dots, t_n$ $n+1$ indeterminates. Let d_j be $n+1$ non-negative integers and let $P_j = \sum_{|\alpha|=d_j} X_{j,\alpha} \mathbf{t}^\alpha$ be $n+1$ homogeneous polynomials of degree d_j in the variables t_0, \dots, t_n generating an ideal I .

Let us write $B := \mathbb{Z}[X_{j,\alpha} \mid j=0, \dots, n, |\alpha|=d_j]$, and $R := B[t_0, \dots, t_n]$. Also let us call S the affine spectrum of B , that is $S := \text{Spec}(B)$. With the assumption $\deg(t_i) = 1$, we have that R/I is a \mathbb{Z} -graded B -algebra. So we can consider the projective S -scheme $Z := \text{Proj}(R/I) \hookrightarrow \mathbb{P}_S^n$, the incidence variety:

$$Z = \{(\mathbf{X}, \mathbf{t}) \in \mathbb{P}^n \times S \mid P_j(\mathbf{X}, \mathbf{t}) = 0, \forall j\} \xrightarrow{i} \mathbb{P}_S^n.$$

We denote by \mathfrak{A} the kernel of the following canonical map of rings $\mathfrak{A} := \ker(B \rightarrow \Gamma(Z, \mathcal{O}_Z)) \cong (H_m^0(R/I))_0$, and set $T := \text{Spec}(B/\mathfrak{A}) \xrightarrow{j} \text{Spec}(B) = S$. We have the following commutative diagram of schemes:

$$\begin{array}{ccc} Z = \text{Proj}(R/I) & \xhookrightarrow{i} & \mathbb{P}_S^n = \mathbb{P}^n \times_{\mathbb{Z}} S \\ \downarrow p & & \downarrow p \\ T = \text{Spec}(B/\mathfrak{A}) & \xhookrightarrow{j} & S = \text{Spec}(B) \end{array} \quad (3)$$

Remark 4. The scheme S parametrizes sequences of polynomials $(P_0, \dots, P_n) \subset \mathbb{Z}[t_0, \dots, t_n]$. A closed point $x = V(\mathbf{m})$ of S belongs to T if and only if the sequence (P_0, \dots, P_n) of associated polynomials has a common root in \mathbb{P}_k^n for some extension k of S/\mathbf{m} .

Theorem 5. [J.-P. Jouanolou] *With the notation above, the following statements are satisfied:*

- (i) \mathfrak{A} is a principal ideal in B , whose generator will be denoted by $\text{Res}_{\mathbb{Z}}(P_0, \dots, P_n)$.
- (ii) The element $\text{Res}(P_0, \dots, P_n)$ of B is not a zero divisor, and for all $j = 0, \dots, n$ is homogeneous of degree $\prod_{i \neq j} d_i$ with respect to the variable X_j .
- (iii) $\text{Spec}(B/(P_0, \dots, P_n))$ is geometrically irreducible in S . Moreover, for any morphism of commutative rings, $\mathbb{Z}[X_{j,\alpha} \mid |\alpha|=d_j] \rightarrow k$, we define $\text{Res}_k(\epsilon(P_0), \dots, \epsilon(P_n)) := \epsilon(\text{Res}_{\mathbb{Z}}(P_0, \dots, P_n))$ where ϵ is extended to a morphism for B to $k[t_0, \dots, t_n]$, linearly in the t_i 's.
- (iv) if k is a field, $\text{Res}_k(\epsilon(P_0), \dots, \epsilon(P_n)) = 0$ if and only if $\epsilon(P_0), \dots, \epsilon(P_n)$ have a common root (different from zero) in an extension of k .

The original presentation of this result in these terms is in (Jou91, Prop 2.3).

Remark 6. If the sequence $\{P_0, \dots, P_n\}$ is regular, $\text{Res}(P_0, \dots, P_n)$ can be computed as the determinant of $\mathcal{K}_\bullet(P_0, \dots, P_n; R)_\nu$, for $\nu > \eta := \sum(d_i - 1)$ (cf. (De84) or (Cha93b)).

3. The algebraic approach

We will establish here the relation between approximation complexes, tensor products of them, and some Koszul complex we will present below.

Assume that the sequence $\{f_i, g_i\}$ is regular for all $i \in \{0, \dots, n\}$. Then, as all $n + 1$ Koszul complexes are acyclic, we have isomorphisms between A and the respective modules of cycles, as the mentioned in the previous section, in Proposition 3.

Definition 7. Write \mathcal{K}_\bullet the Koszul complex associated to the $n + 1$ polynomials $L_i := f_i y_i - g_i x_i$. Denote by \mathcal{Z}_\bullet the complex obtained by tensoring the $n + 1$ approximation complexes $\mathcal{Z}_\bullet(f_i, g_i)$ over A , namely $\mathcal{Z}_\bullet := \bigotimes_{i=0}^n \mathcal{Z}_\bullet(f_i, g_i)$.

Proposition 8. *With the notation above, if the sequences $\{f_i, g_i\}$ are regular for all $i \in \{0, \dots, n\}$, then there exists an isomorphism of A -complexes $\mathcal{Z}_\bullet \cong \mathcal{K}_\bullet$.*

Proof. It is enough to see that by definition of \mathcal{Z}_\bullet and \mathcal{K}_\bullet , and Proposition 3, we have

$$\mathcal{Z}_\bullet := \bigotimes_{i=0}^n \mathcal{Z}_\bullet(f_i, g_i) \cong \bigotimes_{i=0}^n \mathcal{K}_\bullet(L_i; A[x_i, y_i]) \cong \mathcal{K}_\bullet \quad \square$$

From Proposition 8 we deduce

Corollary 9. *We can resolve the algebra*

$$\mathcal{B} := \bigotimes_A \text{Sym}_{(f_i, g_i)}(A[x_i, y_i]) \cong \frac{A[\mathbf{X}]}{(f_i y_i - g_i x_i)_{i=0, \dots, n}}, \quad (4)$$

by means of the \mathcal{Z}_\bullet complex, or, equivalently when the hypothesis of Proposition 8 are satisfied, by means of the Koszul complex \mathcal{K}_\bullet .

We have the following result.

Lemma 10. *The \mathcal{Z}_\bullet complex satisfies*

- (i) $H_0(\mathcal{Z}_\bullet) \cong H_0(\mathcal{K}_\bullet) \cong \frac{A[\mathbf{X}]}{(f_i y_i - g_i x_i)_{i=0, \dots, n}}$, and
- (ii) \mathcal{Z}_\bullet is acyclic if and only if the Koszul complex $\mathcal{K}_\bullet(L_0, \dots, L_n; A[\mathbf{X}])$ is, that is, if and only if the sequence L_0, \dots, L_n is regular.

Remark 11. We said that Avramov's criterion, stated in Theorem 1, gives a necessary and sufficient condition for the acyclicity of \mathcal{K}_\bullet for being acyclic. That is, conditions on \mathcal{B} for being a complete intersection.

As mentioned in Section 2.3, the resultant of L_0, \dots, L_n as polynomials in the variables t_0, \dots, t_n , can be computed as a MacRae invariant of $(R/I)_\nu$, for $\nu > \sum(d_i - 1)$, which is the determinant of a suitable resolution of R/I in degree ν . Observe that these two complexes are naturally bigraded, one grading corresponds to the \mathbf{X} variables, and the other to the \mathbf{t} variables. The acyclicity condition on \mathcal{Z}_\bullet is applied to the first group, and the notation $(R/I)_\nu$ stands for the grading on the second group. Hence, for a fixed ν , $(\mathcal{Z}_\bullet)_\nu$ is a resolution of the $k[\mathbf{X}]$ -module of $(R/I)_\nu$.

Consequently we get by this method a multi-homogeneous generator of the ideal $\mathfrak{A} \subset k[\mathbf{X}]$, that is, we have the following implicitization result:

Corollary 12. *If the sequence L_0, \dots, L_n is regular, then the complex*

$$(\mathcal{K}_\bullet(L_0, \dots, L_n; A[\mathbf{X}]))_\nu : 0 \rightarrow (\mathcal{Z}_{n+1})_\nu \xrightarrow{\partial_{n+1}} \dots \xrightarrow{\partial_2} (\mathcal{Z}_1)_\nu \xrightarrow{\partial_1} (\mathcal{Z}_0)_\nu \rightarrow 0$$

is acyclic for all $\nu > \eta := \sum(d_i - 1)$.

Moreover, with the notation of Section 2.3, $\det((\mathcal{K}_\bullet(L_0, \dots, L_n; A[\mathbf{X}]))_\nu)$ gives an implicit equation for the closed image of $p : Z = \text{BiProj}(R/I) \rightarrow \tilde{S} = \text{Proj}(B)$.

Proof. The result follows from Remark 6 and Theorem 5. For instance, by the universality of the resultant (cf. Theorem 5 parts (iii) and (iv)) it is enough to see this for generic coefficients. By Remark 6 $\det((\mathcal{K}_\bullet(L_0, \dots, L_n; A[\mathbf{X}]))_\nu)$ computes $\text{Res}_A(L_0, \dots, L_n)$ for $\nu > \sum(d_i - 1)$. Again, by Theorem 5, this resultant computes the divisor obtained as the projection of the incidence scheme according to Diagram (3) in the projective context:

$$\begin{array}{ccc} Z = \text{BiProj}(R/I) & \xhookrightarrow{i} & \mathbb{P}^n_{\tilde{S}} = \mathbb{P}^n \times_{\mathbb{Z}} \tilde{S} \\ \downarrow p & & \downarrow p \\ T = \text{Proj}(B/\mathfrak{A}) & \xhookrightarrow{j} & \tilde{S} = \text{Proj}(B). \end{array}$$

The results follows from the fact that the image of this projection is the closed image of $p : Z = \text{BiProj}(R/I) \rightarrow \tilde{S} = \text{Proj}(B)$. \square

In the next section we will focus on the geometric interpretation of this fact, and in reinterpreting this result in terms of the scheme theoretic image of ϕ .

4. The geometric approach

4.1. The implicit equation as a resultant

In this section we will focus on the geometrical aspects related to the acyclicity of the Koszul complex \mathcal{K}_\bullet , and the nature of the base locus of the rational map $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$. Recall ϕ is defined by $(t_0 : \dots : t_n) \mapsto (f_0 : g_0) \times \dots \times (f_n : g_n)$ where f_i, g_i are homogeneous polynomials of degree d_i . Write $I^{(i)} := (f_i, g_i)$ the homogeneous ideal of A defined by f_i, g_i .

From Proposition 8, the regularity of the sequence $\{f_i, g_i\}$ for all i implies that the complex \mathcal{K}_\bullet coincides with the tensor product of the approximation complex of cycles, \mathcal{Z}_\bullet (cf. Definition 7). Hence, we will focus on the conditions on \mathcal{K}_\bullet for being acyclic, and on its geometrical interpretation.

As we mentioned above, we want here to use a suitable complex for computing $\mathfrak{A} := \text{Res}_A(L_0, \dots, L_n)$. Also we will explain that under certain conditions on the codimension of some ideals of minors of the matrix M defined in (1), we can assure that this resultant is exactly an implicit equation of the scheme theoretic image of ϕ to a certain power, $\deg(\phi)$.

Write X for the closed subscheme of \mathbb{P}^n , given by the common zeroes of all $2(n+1)$ polynomials, and denote by W the base locus. That is:

$$X := \text{Proj}(A / \sum_i I^{(i)}), \text{ and } W := \text{Proj}(A / \prod_i I^{(i)}) \quad (5)$$

Set-theoretically we have that $X = \bigcap_{i=0}^n V(f_i, g_i)$ and $W = \bigcup_{i=0}^n V(f_i, g_i)$ inside \mathbb{P}^n . Clearly, we always have $X \subset W$.

Recall that given a matrix M as described in (1), we write $I_r = I_r(M)$ for the ideal of A generated by the $r \times r$ minors of M , for $0 \leq r \leq r_0 := \min\{n+1, m\}$, where $I_0 := A$ and $I_r = 0$ for $r > r_0$. From Theorem 1 we have a condition for the acyclicity of the Koszul complex \mathcal{K}_\bullet in terms of the ideals I_r .

For instance, as A is a Cohen Macaulay ring, codimension coincides with depth. In particular, when $n = 2$, we have, from 1, that $\text{codim}_{\mathbb{P}^2}(V(I_1)) \geq 3$, this is $X := V(I_1) = V(I^{(0)} + I^{(1)} + I^{(2)}) = \emptyset$, and that $\text{codim}_{\mathbb{P}^2}(V(I_2)) \geq 2$, which implies that $V(I_2) = \{p_1, \dots, p_s\}$ is a finite set. As $V(I_3) = W$ is the base locus, $\text{codim}_{\mathbb{P}^2}(V(I_3)) \geq 1$ is satisfied when the base points of ϕ have codimension less than or equal to 1. More generally:

Theorem 13. *Let $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$ be, as in the previous section, a finite rational map given by $(t_0 : \dots : t_n) \mapsto (f_0, g_0) \times \dots \times (f_n, g_n)$ where f_i, g_i are homogeneous polynomials of degree d_i , and for all $i = 0, \dots, n$*

not both f_i, g_i are the zero polynomial. Let us denote by A the polynomial ring $k[t_0, \dots, t_n]$, with L_i the expression $f_i y_i - g_i x_i$, for $i = 0, \dots, n$, and with $I^{(i)}$ the ideal $(f_i, g_i) \subset A$.

(i) The following statements are equivalent:

(a) the Koszul complex \mathcal{K}_\bullet is a free resolution of $\mathcal{B} := \frac{A[x_0, y_0, \dots, x_n, y_n]}{(L_0, \dots, L_n)}$.

(b) $\text{codim}_A(I_r) \geq (n+1) - r + 1$ for all $r = 1, \dots, n+1$.

(c) all the following statements are true:

(i) $\bigcap_{i=1}^n V(I^{(i)}) = \emptyset$.

(ii) $\#(\bigcap_{i < j} (V(I^{(i)} \cdot I^{(j)}))) < \infty$.

(iii) $\dim(\bigcap_{i < j < k} (V(I^{(i)} \cdot I^{(j)} \cdot I^{(k)}))) \leq 1$.

\vdots

(n) $\dim(\bigcap_i (V(I^{(0)} \cdot \dots \cdot I^{(i)} \cdot \dots \cdot I^{(n)}))) \leq n - 2$.

(ii) If any (all) of the items before are satisfied, then:

(a) The (multi)homogeneous resultant $\text{Res}_{A, d_0, \dots, d_n}(L_0, \dots, L_n)$ is not the zero polynomial in A .

(b) Denote by H the irreducible implicit equation of the scheme theoretic image \mathcal{H} of ϕ . If for all $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$ we have that $\text{codim}_A(I^{(i_0)} + \dots + I^{(i_k)}) > k + 1$, then, for $\nu > \eta = \sum_i (d_i - 1)$,

$$\det((\mathcal{K}_\bullet)_\nu) = \text{Res}_A(L_0, \dots, L_n) = H^{\deg(\phi)}.$$

Proof. (i)(a) \Leftrightarrow (i)(b) follows from Avramov's Theorem 1.

(i)(b) \Leftrightarrow (i)(c) Note that each $r \times r$ -minor of M can be expressed as a product of r polynomials, where for each column we choose either f or g . Then, the ideal of minors involving the columns i_0, \dots, i_{r-1} coincides with the ideal $I^{(i_0)} \dots I^{(i_{r-1})}$. As we assumed that for any i $f_i \neq 0$ and $g_i \neq 0$, the condition $\dim(V(I^{(0)} \dots I^{(n)})) \leq n - 1$ is automatically satisfied.

(i)(a) \Rightarrow (ii)(a) this is a classical result first studied by J.-P. Jouanolou in (Jou95, §3.5), reviewed in (GKZ94), and also used by L. Busé, M. Chardin and J.-P. Jouanolou, in their previous work in the area.

(ii)(b) Let us denote by

$$Z = \{(t, x) : t = (t_0, \dots, t_n), x = (x_0, y_0, \dots, x_n, y_n), L_i(t, x) = 0, \forall i = 0, \dots, n\}.$$

As all the polynomials L_i are multihomogeneous (and so homogeneous) in the variables x_i, y_i , and homogeneous in the t_i , then we can think Z as the incidence subvariety in $\mathbb{P}^n \times (\mathbb{P}^1)^{n+1}$, and in $\mathbb{P}^n \times \mathbb{P}^{2(n+1)-1}$. We will return to this fact to reduce the problem of analysing the homogeneous resultant in the space $\mathbb{P}^{2(n+1)-1}$.

Write Z_Ω for the open set defined by the points $z \in Z$, such that $\pi_1(z) \in \Omega := \mathbb{P}^n - W$, where W is the base locus, as in (5). The closed subscheme Z of $\mathbb{P}^n \times (\mathbb{P}^1)^{n+1}$ corresponds to the projective scheme $\text{MultiProj}(\mathcal{B})$ and Z_Ω is the open subset of Z that is isomorphic to the complement of the base locus on \mathbb{P}^n . For p in the base locus of ϕ , e.g. $p \in V(I^{(i)})$, there is a commutative diagram of schemes as follows:

$$\begin{array}{ccccc} & & Z_\Omega & \hookrightarrow & Z & \longrightarrow & \mathbb{P}^n \times (\mathbb{P}^1)^{n+1} \\ & & \downarrow & & \downarrow \pi_1 & \searrow \pi_2 & \\ \pi_1^{-1}(p) & & & & & & \\ & & \Omega & \hookrightarrow & \mathbb{P}^n & \xrightarrow{\phi} & (\mathbb{P}^1)^{n+1} \\ & \nearrow & & & \uparrow & & \\ \{p\} & \hookrightarrow & V(I^{(i)}) & & & & \end{array}$$

Observe that the closed subscheme Z is the Zariski closure of Z_Ω , and that the closure of the second projection of Z_Ω coincides with the scheme theoretic image of ϕ , this is $\pi_2(\overline{Z_\Omega}) = \mathcal{H}$. Assume that $\text{Res}_{A, d_0, \dots, d_n}(L_0, \dots, L_n) \neq 0$, hence, this equation defines a divisor $[\pi_2(Z)]$ in $(\mathbb{P}^1)^{n+1}$.

Due to the multihomogeneity of the resultant, the polynomial $\text{Res} := \text{Res}_{A, d_0, \dots, d_n}(L_0, \dots, L_n) \in k[\mathbf{X}]$ is multihomogeneous, hence it defines a closed subscheme in $(\mathbb{P}^1)^{n+1}$. By Theorem 5, this element is

the image via the specialization map $\epsilon : \mathbb{Z}[\text{coef}(f_i, g_i) : i = 0, \dots, n] \rightarrow k$, of the irreducible equation $\text{Res}_{\mathbb{Z}, d_0, \dots, d_n}(L_0, \dots, L_n) \in \mathbb{Z}[\text{coef}(f_i, g_i), \forall i][\mathbf{X}]$.

We claim that our hypotheses are the necessary ones to avoid extra factors. For this, set $\alpha := i_0, \dots, i_k$, and write

$$X_\alpha := \text{Proj}(A / \sum_{j=0}^k I^{(i_j)}), \quad X_j := X_{\{j\}} \text{ for just one } j, \text{ and } U_\alpha := X_\alpha - \bigcup_{j \notin \alpha} X_j. \quad (6)$$

Observe that X_α stands for the subset of W , containing X (cf. (5)), where the equations L_{i_0}, \dots, L_{i_k} vanish identically. If U_α is non-empty, consider $p \in U_\alpha$, then $\dim(\pi_1^{-1}(p)) = |\alpha| = k + 1$. As the fibre is equidimensional, write $\mathcal{E}_\alpha := \pi_1^{-1}(U_\alpha) \subset \mathbb{P}^n \times (\mathbb{P}^1)^{n+1}$ for the the fibre over U_α , which defines a (multi)projective bundle of rank $|\alpha|$. Hence, we have

$$\text{codim}(\mathcal{E}_\alpha) = n + 1 - (k + 1) + (\text{codim}_{\mathbb{P}^n}(U_\alpha)).$$

The condition $\text{codim}_A(I^{(i_0)} + \dots + I^{(i_k)}) > k + 1$, for all $\{i_0, \dots, i_k\} \subset \{1, \dots, n\}$ implies that $\text{codim}_{\mathbb{P}^n}(U_\alpha) > k + 1$. Hence

$$\text{codim}(\mathcal{E}_\alpha) > n + 1 = \text{codim}(Z_\Omega).$$

Observe that $Z - Z_U = Z_\Omega$, where $Z_U := \bigcap_\alpha \mathcal{E}_\alpha$, and that $\text{codim}(Z_U) > n + 1 = \text{codim}(Z_\Omega) = \text{codim}(\overline{Z_\Omega})$.

As $\text{Spec}(B)$ is a complete intersection, in $\mathbb{A}^{2(n+1)}$ it is unmixed and purely of codimension $n + 1$. Thus, $Z \neq \emptyset$ is also purely of codimension $n + 1$. This and the fact that $\text{codim}(Z_U) > n + 1$ implies that $Z = \overline{Z_\Omega}$.

The graph Z_Ω is irreducible, hence Z is irreducible as well, and its projection (the closure of the image of ϕ) is of codimension 1, and one has for $\nu > \eta$:

$$[\det((\mathcal{K}_\bullet)_\nu)] = \text{div}_{k[\mathbf{X}]}(H_0(\mathcal{K}_\bullet)_\nu) = \text{div}_{k[\mathbf{X}]}(\mathcal{B}_\nu) = \sum_{\mathbf{p} \text{ prime, } \text{codim}_{k[\mathbf{X}]}(\mathbf{p})=1} \text{length}_{k[\mathbf{X}]_{\mathbf{p}}}((\mathcal{B}_\nu)_{\mathbf{p}})[\mathbf{p}].$$

Also as $[\det((\mathcal{K}_\bullet)_\nu)] = \text{div}_{k[\mathbf{X}]}(\text{Res}) = e \cdot [\mathbf{q}]$ for some integer e and $\mathbf{q} := (H) \subset k[\mathbf{X}]$, we have that $\sum_{\mathbf{p} \text{ prime, } \text{codim}(\mathbf{p})=1} \text{length}_{k[\mathbf{X}]_{\mathbf{p}}}((\mathcal{B}_\nu)_{\mathbf{p}})[\mathbf{p}] = e \cdot [\mathbf{q}]$ hence $[\det((\mathcal{K}_\bullet)_\nu)] = \text{length}_{k[\mathbf{X}]_{\mathbf{q}}}((\mathcal{B}_\nu)_{\mathbf{q}})[\mathbf{q}]$. Finally for $\nu \gg 0$

$$\text{length}_{k[\mathbf{X}]_{\mathbf{q}}}((\mathcal{B}_\nu)_{\mathbf{q}}) = \dim_{\kappa(\mathbf{q})}(\mathcal{B}_\nu \otimes_{k[\mathbf{X}]_{\mathbf{q}}} \kappa(\mathbf{q})) = \deg(\phi),$$

where $\kappa(\mathbf{q}) := k[\mathbf{X}]_{\mathbf{q}}/\mathbf{q}k[\mathbf{X}]_{\mathbf{q}}$. This shows that $e = \deg(\phi)$ and completes the proof. \square

Remark 14. We showed that the scheme $\pi_2(Z)$ is defined by the polynomial Res , while the closed image of ϕ coincides with $\pi_2(\overline{Z_\Omega})$, hence the polynomial H divides Res . Moreover, from the proof above we conclude that $H^{\deg(\phi)}$ also divides Res . And if $[\mathcal{E}_\alpha]$ is an algebraic cycle of $\mathbb{P}^n \times (\mathbb{P}^1)^{n+1}$ of codimension $n + 1$, then $[\pi_2(\mathcal{E}_\alpha)]$ is not a divisor in $(\mathbb{P}^1)^{n+1}$, and consequently Res has no other factor than $H^{\deg(\phi)}$.

Remark 15. With the hypotheses of Theorem 13(ii)(b), denoting by \deg_i the degree on the variables x_i, y_i and by \deg_{tot} the total one, we have:

- (i) $\deg_i(H) \deg(\phi) = \prod_{j \neq i} d_j$;
- (ii) $\deg_{\text{tot}}(H) \deg(\phi) = \sum_i \prod_{j \neq i} d_j$.

In the rest of this section we will focus on the study of the difference between the closed schemes $\overline{Z_\Omega}$ and Z , and its projection $\pi_2(Z - \overline{Z_\Omega})$ on $(\mathbb{P}^1)^{n+1}$. We will show how this can give extra factors on the resultant.

4.2. Analysis of the extra factors

In this section our aim is to analyse in detail the nature of the extra factors that appear when the codimension condition of Theorem 13(ii)(b) does not hold. We will see that these polynomials appear due to the existence of big enough dimensional fibres over some of the components of the base locus. In a few words, it happens that if the sum of the dimension of a component in the base locus plus the dimension of its fibre is n , there is a “big dimensional” subscheme whose projection on $(\mathbb{P}^1)^{n+1}$ defines a hypersurface which gives rise to a factor in the computed resultant.

In order to understand this, we will first analyse some simple cases, namely, where this phenomenon occurs over a finite set of points of the base locus; and later, we will deduce the general implicitization result.

Example 16. Assume we are given a rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where ϕ maps $(u : v : w)$ to $(f_0 : g_0) \times (f_1 : g_1) \times (f_2 : g_2)$, of degrees d, d' and d'' respectively.

We may suppose that each of the pairs of polynomials $\{f_0, g_0\}$, $\{f_1, g_1\}$ and $\{f_2, g_2\}$ have no common factors. Then, the condition $\text{codim}_A(I^{(i)}) \geq 2$ is automatically satisfied. Assume also that $X = \emptyset$, this is, there are no common roots to all 6 polynomials.

We will show here that, if we don't ask for the "correct" codimension conditions, we could be implicitizing some extra geometric objects. For instance, suppose that we take a simple point $p \in V(I^{(0)} + I^{(1)}) \neq \emptyset$. Consequently $L_0(u, v, w, \mathbf{X}) = L_1(u, v, w, \mathbf{X}) = 0$ for all choices of \mathbf{X} . Nevertheless, $L_2(u, v, w, \mathbf{X}) = g_2(u, v, w)x_2 - f_2(u, v, w)y_2 = 0$ imposes the nontrivial condition $g_2(p)x_2 - f_2(p)y_2 = 0$ on (x_2, y_2) , hence there is one point $q = (f_2(p) : g_2(p)) \in \mathbb{P}^1$ which is the solution of this equation. We get $\pi_1^{-1}(p) = \{p\} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \{q\}$. As we do not want the reader to focus on the precise computation of this point q , we will usually write $\{*\}$ for the point $\{q\}$ obtained as the solution of the only nontrivial equation.

Suppose also that, for simplicity, $V(I^{(0)} + I^{(1)}) = \{p\}$, $V(I^{(0)} + I^{(2)}) = \emptyset$, and $V(I^{(1)} + I^{(2)}) = \emptyset$. This says that if we compute $\pi_2(Z)$, then we get

$$\begin{aligned} \pi_2(Z) &= \pi_2(\pi_1^{-1}(\overline{\Omega \cup W})) = \pi_2(\overline{\pi_1^{-1}(\Omega)}) \cup \pi_2(\pi_1^{-1}(W)) = \\ &= \pi_2(\overline{Z_\Omega}) \cup (\pi_2(\{p\} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \{*\}) = \\ &= \overline{\text{im}(\phi)} \cup (\mathbb{P}^1 \times \mathbb{P}^1 \times \{*\}), \end{aligned}$$

where $W = \text{Proj}(A / \prod_i I^{(i)})$ the base locus of ϕ as in (5), and $\Omega = \mathbb{P}^n - W$ its domain.

Hence, the Macaulay Resultant contains some extra factor. Let us observe that if there is only one extra hyperplane appearing (over a point p with multiplicity one), which corresponds to $\pi_2(\pi_1^{-1}(p))$, then $\pi_1^{-1}(p)$ is a closed subscheme of Z , defined by the equation $L_3(p) = 0$. Then, we will show that

$$\text{Res}_{(u,v,w)}(L_0, L_1, L_2) = \mathcal{H}^{\deg(\phi)} \cdot L_3(p).$$

We will now generalize Theorem 13 in the spirit of the example above. For each $i \in \{0, \dots, n\}$ take $X_i := \text{Proj}(A / \sum_{j \neq i} I^{(j)})$.

Proposition 17. *Let $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$ be a rational map that satisfies conditions a – c of Theorem 13. Assume further that for all $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$, with $k < n - 1$, $\text{codim}_A(I^{(i_0)} + \dots + I^{(i_k)}) > k + 1$. Then, there exist non-negative integers μ_p such that:*

$$\text{Res}_A(L_0, \dots, L_n) = H^{\deg(\phi)} \cdot \prod_{i=0}^n \prod_{p \in X_i} L_i(p)^{\mu_p}.$$

Proof. Denote by $\Gamma := \overline{Z_\Omega}$ the closure of the graph of ϕ , Z as before. From Remark 14, we can write

$$G := \frac{\text{Res}_A(L_0, \dots, L_n)}{H^{\deg(\phi)}},$$

the extra factor. It is clear that G defines a divisor in $(\mathbb{P}^1)^{n+1}$ with support on $\pi_2(Z - \Gamma)$. From the proof of Theorem 13, we have that Z and Γ coincide outside $W \times (\mathbb{P}^1)^{n+1}$. As Z is defined by linear equations in the second group of variables, then $Z - \Gamma$ is supported on a union of linear spaces over the points of W , and so, its closure is supported on the union of the linear spaces $(\pi_1)^{-1}(p) \cong \{p\} \times ((\mathbb{P}^1)^n \times \{*\})$, where $\{*\}$ is the point $(x : y) \in \mathbb{P}^1$ such that $L_i(p, x, y) = 0$ for suitable i . It follows that $\pi_2((\pi_1)^{-1}(p)) \subset V(L_i) \subset (\mathbb{P}^1)^{n+1}$, and consequently

$$G = \prod_{p \in X} L_i(p)^{\mu_p},$$

for some non-negative integers μ_p . □

As we said at the begining of this section, we are interested in understanding the extra factors. We have shown above that these factors depend directly on the dimension of each component of the base locus and on the dimension of its fibre; and that if their sum equals n this generates an extra hypersurface, hence an extra factor. It is natural to ask, why this sum does not exceed n . The next lemma gives an answer to this question.

Lemma 18. *Let $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$, be a finite rational map as above. Under the hypothesis of Theorem 1, $\text{codim}_A(\sum_{j=0}^k I^{(i_j)}) \geq k + 1$ for all $k = 0, \dots, n$ and $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$.*

Proof. To show this we will use Theorem 1. Let us denote by I the ideal $I^{(i_0)} + \dots + I^{(i_n)}$, and for a fixed k write $I = I' + I''$, where $I' = \sum_{j=0}^k I^{(i_j)}$ and $I'' = \sum_{l=k+1}^n I^{(i_l)}$ the sum over the complementary indices of $\{i_0, \dots, i_k\}$. As (L_0, \dots, L_n) is a complete intersection in R , also is $(L_{i_0}, \dots, L_{i_k})$ in $A[x_{i_0}, y_{i_0}, \dots, x_{i_k}, y_{i_k}]$. Applying Theorem 1 to the ideal $(L_{i_0}, \dots, L_{i_k})$, for $r = 1$ we have that $\text{codim}_A(I^{(i_0)} + \dots + I^{(i_k)}) \geq k + 1$.

Observe that as I' is generated by a subset of the set of generator of I then I' is also a complete intersection in R . Now, as it is generated by elements only depending on the variables x_{i_j}, y_{i_j} for $j = 0, \dots, k$, we have that it is also a complete intersection in $A[x_{i_0}, y_{i_0}, \dots, x_{i_k}, y_{i_k}]$. \square

Remark 19. It is easy to see that the converse of Lemma 18 does not always hold (in fact almost never). For example let us take for $n = 1$, $f_0 = g_0 = f_1 = g_1 \neq 0$.

We will now introduce some notation we will use to state and prove Theorem 22. The idea behind these definitions is to split the base locus into irreducible subvarieties with the desired dimension, in order to obtain a clear factorization of the extra component that appears.

Definition 20. For each $\alpha := (i_0, \dots, i_k)$ set $I^\alpha := \sum_{j=0}^k I^{(i_j)}$, and $X_\alpha := \text{Proj}(A/I^\alpha)$ as defined in (6). Denote by Θ the set of all $\alpha \subset \{0, \dots, n\}$ such that $\text{codim}(I^\alpha) = |\alpha|$.

For each $\alpha := (i_0, \dots, i_k) \in \Theta$, let $I^\alpha = (\cap_{\mathbf{q}_i \in \Lambda_\alpha} \mathbf{q}_i) \cap \mathbf{q}'$, where Λ_α is the set of primary ideals of codimension $|\alpha|$ containing I , and $\text{codim}_A(\mathbf{q}') > |\alpha|$. Write

$$X_{\alpha,i} := \text{Proj}(A/\mathbf{q}_i), \quad \text{with } \mathbf{q}_i \in \Lambda_\alpha, \quad (7)$$

and let $X_{\alpha,i}^{\text{red}}$ be the associated reduced variety. Denote by π_α the projection onto the coordinates $(x_i : y_i)$, for $i \notin \alpha$. Namely, let $\{i_{k+1}, \dots, i_n\} := \{1, \dots, n\} - \alpha$, then π_α is given by

$$\begin{aligned} \pi_\alpha : (\mathbb{P}^1)^{n+1} &\rightarrow (\mathbb{P}^1)^{n+1-|\alpha|} \\ (x_0 : y_0) \times \dots \times (x_n : y_n) &\mapsto (x_{i_{k+1}} : y_{i_{k+1}}) \times \dots \times (x_{i_n} : y_{i_n}). \end{aligned} \quad (8)$$

Set $\phi_\alpha : \mathbb{P}^n \dashrightarrow \mathbb{P}_\alpha := \pi_\alpha((\mathbb{P}^1)^{n+1})$ defined as $\phi_\alpha := \pi_\alpha \circ \phi$. Hence, ϕ_α is explicitly defined as $\phi_\alpha(\mathbf{t}) = (f_{i_{k+1}}(\mathbf{t}) : g_{i_{k+1}}(\mathbf{t})) \times \dots \times (f_{i_n}(\mathbf{t}) : g_{i_n}(\mathbf{t}))$ as is shown in the diagram below

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\phi} & (\mathbb{P}^1)^{n+1} \\ & \searrow \phi_\alpha & \downarrow \pi_\alpha \\ & & \mathbb{P}_\alpha := \pi_\alpha((\mathbb{P}^1)^{n+1}) \cong (\mathbb{P}^1)^{n+1-|\alpha|}. \end{array}$$

Denote by W_α the base locus of ϕ_α and $X := \text{Proj}(A/\sum_i I^{(i)})$. Since $W := \text{Proj}(A/\prod_i I^{(i)})$ is the base locus of ϕ as defined in (5), we have the inclusions $X \subset W_\alpha \subset W$.

Denote by $\mathcal{U}_\alpha := \mathbb{P}^n - W_\alpha$, the open set where ϕ_α is well defined. Write $\Omega_\alpha := X_\alpha \cap \mathcal{U}_\alpha$ and $\Omega_{\alpha,i} := X_{\alpha,i} \cap \mathcal{U}_\alpha$. The open set $\Omega_{\alpha,i}$ stands for the points of \mathbb{P}^n which give rise to an extra factor. This factor coincides with the restricted image of ϕ over $\Omega_{\alpha,i}$ to a certain power.

If α is empty, we set $\pi_\alpha = \text{Id}_{(\mathbb{P}^1)^{n+1}}$, $\phi_\alpha = \phi$, $W_\alpha = W$ and $\mathcal{U}_\alpha = \Omega_\alpha = \Omega$.

Next, we show that this meticulous decomposition of the base locus has the properties we are looking for. Namely, the fibre over each of the open sets $\Omega_{\alpha,i}$ has dimension $|\alpha|$, in such a way that the sum of the dimensions of the base plus the dimension of the fiber is n .

Lemma 21. *Let ϕ be as in Theorem 13. For each $\alpha \in \Theta$ and each $\mathbf{q}_i \in \Lambda_\alpha$, the following statements are satisfied:*

- (i) $\Omega_{\alpha,i}$ is non-empty.
- (ii) for all $p \in \Omega_{\alpha,i}$, $\dim(\pi_1^{-1}(p)) = |\alpha|$.
- (iii) the restriction $\phi_{\alpha,i}$ of ϕ to $\Omega_{\alpha,i}$, defines a rational map

$$\phi_{\alpha,i} : X_{\alpha,i} \dashrightarrow \mathbb{P}_\alpha \cong (\mathbb{P}^1)^{n+1-|\alpha|}. \quad (9)$$

- (iv) $Z_{\alpha,i} := \pi_1^{-1}(\Omega_{\alpha,i})$ is a (multi)projective bundle $\mathcal{E}_{\alpha,i}$ of rank $|\alpha|$ over $\Omega_{\alpha,i}$.

Proof. Fix $X_{\alpha,i} \subset X_\alpha$ and write $\alpha := i_0, \dots, i_k$. Note that $\Omega_{\alpha,i} := X_{\alpha,i} - \bigcup_{j \notin \alpha} X_{\{j\}}$ is an open subset of $X_{\alpha,i}$. If $\Omega_{\alpha,i} = \emptyset$, then $X_{\alpha,i} \subset \bigcup_{j \notin \alpha} X_{\{j\}}$, and since it is irreducible, there exists j such that $X_{\alpha,i} \subset X_{\{j\}}$, hence $X_{\alpha,i} \subset X_{\{j\}} \cap X_\alpha = X_{\alpha \cup \{j\}}$. Setting $\alpha' := \alpha \cup \{j\}$, it follows that $\dim(X_{\alpha'}) \geq \dim(X_{\alpha,i}) = n - |\alpha| > n - |\alpha'|$, which contradicts the hypothesis.

Let $p \in \Omega_{\alpha,i}$, $\pi_1^{-1}(p) = \{p\} \times \{q_{i_{k+1}}\} \times \dots \times \{q_{i_n}\} \times (\mathbb{P}^1)^{|\alpha|}$, where the point $q_{i_j} \in \mathbb{P}^1$ is the only solution to the nontrivial equation $L_{i_j}(p, x_{i_j}, y_{i_j}) = y_{i_j} f_{i_j}(p) - x_{i_j} g_{i_j}(p) = 0$. Then we deduce that $\dim(\pi_1^{-1}(p)) = |\alpha|$, and that $\phi_{\alpha,i} : \Omega_{\alpha,i} \rightarrow \prod_{j=k+1}^n \mathbb{P}_{(i_j)}^1 \cong (\mathbb{P}^1)^{n+1-|\alpha|}$ given by $p \in \Omega_{\alpha,i} \mapsto \{q_{i_{k+1}}\} \times \dots \times \{q_{i_n}\} \in \prod_{j=k+1}^{n+1} \mathbb{P}_{(i_j)}^1$, is well defined.

The last statement follows immediately from the previous ones. \square

Note that $X_{\alpha,i}$ has dimension $n - |\alpha|$, by the preceding lemma $\dim(\pi_1^{-1}(\Omega_{\alpha,i})) = n$. Since $\text{Res}_A(L_0, \dots, L_n)$ describes the codimension one part of $\pi_2(Z)$, if $\dim(\pi_2(\pi_1^{-1}(\Omega_{\alpha,i}))) = n$, then $\text{Res}_A(L_0, \dots, L_n)$ is not irreducible. Denote by $\Delta_\alpha \subset \Lambda_\alpha$ the subset of primary ideals \mathbf{q}_i satisfying $\dim(\pi_2(\pi_1^{-1}(X_{\alpha,i}))) = n$. Observe that if $|\alpha| = n$ we are in the case of Proposition 17. The following diagram illustrates this situation:

$$\begin{array}{ccccc}
 & & Z & \xrightarrow{\quad} & \mathbb{P}^n \times (\mathbb{P}^1)^{n+1} \\
 & \nearrow & \downarrow \pi_1 & \searrow \pi_2 & \\
 \pi_1^{-1}(p) & \hookrightarrow & \mathcal{E}_{\alpha,i} & & \\
 \downarrow & & \downarrow & & \\
 \{p\} & \hookrightarrow & \Omega_{\alpha,i} & \xrightarrow{\quad} & \mathbb{P}^n \xrightarrow{\quad \phi \quad} (\mathbb{P}^1)^{n+1} \\
 & \searrow & \downarrow & \nearrow \phi_{\alpha,i} & \downarrow \pi_\alpha \\
 & & X_{\alpha,i} & \xrightarrow{\quad \phi_{\alpha,i} \quad} & \pi_2(\mathcal{E}_{\alpha,i}) \hookrightarrow (\mathbb{P}^1)^{n+1-|\alpha|}
 \end{array}$$

where in this case $\pi_1^{-1}(p) = \{p\} \times \{q_{i_{k+1}}\} \times \dots \times \{q_{i_n}\} \times (\mathbb{P}^1)^{|\alpha|}$ is the $|\alpha|$ -dimensional fibre of $\mathcal{E}_{\alpha,i}$ over p , for some $q_{i_j} \in \mathbb{P}^1$.

We finally state our general result.

Theorem 22. *Let $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$ be a finite rational map, satisfying the hypotheses of Theorem 13 and conditions (i.a) – (i.c) in that theorem, hence $\text{Res}_A(L_0, \dots, L_n) \neq 0$. Denote by H the irreducible implicit equation of the closure of the image of ϕ , and by $H_{\alpha,i}$ the irreducible implicit equation of the closure of the image of $\phi_{\alpha,i}$ defined in (9) for each $\alpha \in \Theta$, and i such that $\mathbf{q}_i \in \Delta_\alpha$.*

Then, there exist positive integers $\mu_{\alpha,i}$ such that:

$$\text{Res}_A(L_0, \dots, L_n) = H^{\deg(\phi)} \cdot \prod_{\alpha \in \Theta} \prod_{i: \mathbf{q}_i \in \Delta_\alpha} (H_{\alpha,i})^{\mu_{\alpha,i} \deg(\phi_{\alpha,i})}.$$

Proof. Denote by $\Gamma := \overline{Z_\Omega}$, the graph of ϕ , Z the incidence scheme, and $G = \frac{\text{Res}_A(L_0, \dots, L_n)}{H^{\deg(\phi)}}$ the extra factor.

As in Proposition 17, G defines a divisor in $(\mathbb{P}^1)^{n+1}$ with support on $\pi_2(Z - \Gamma)$, and Z and Γ coincide outside $W \times (\mathbb{P}^1)^{n+1}$.

By part (iii) of Lemma 21, for each α and each $\mathbf{q}_i \in \Delta_\alpha \subset \Lambda_\alpha$, $\phi_{\alpha,i}$ defines a (multi)projective bundle $\mathcal{E}_{\alpha,i}$ of rank $|\alpha|$ over $\Omega_{\alpha,i}$. By definition of Δ_α , $\overline{\pi_2(\mathcal{E}_{\alpha,i})}$ is a closed subscheme of $(\mathbb{P}^1)^{n+1}$ of codimension 1.

Denoting by $[\overline{\mathcal{E}_{\alpha,i}}] = \mu_{\alpha,i} \cdot [\overline{\mathcal{E}_{\alpha,i}^{red}}]$ the class of $\overline{\mathcal{E}_{\alpha,i}}$ as an algebraic cycle of codimension $n+1$ in $\mathbb{P}^n \times (\mathbb{P}^1)^{n+1}$, we have $(\pi_2)_*[\overline{\mathcal{E}_{\alpha,i}}] = \mu_{\alpha,i} \cdot (\pi_2)_*[\overline{\mathcal{E}_{\alpha,i}^{red}}] = \mu_{\alpha,i} \cdot \deg(\phi_{\alpha,i}) \cdot [\mathfrak{p}_{\alpha,i}]$, where $\mathfrak{p}_{\alpha,i} := (H_{\alpha,i})$.

As in Theorem 13, one has for $\nu > \eta$:

$$[\det((\mathcal{K}_{\bullet})_{\nu})] = \operatorname{div}_{k[\mathbf{X}]}(H_0(\mathcal{K}_{\bullet})_{\nu}) = \operatorname{div}_{k[\mathbf{X}]}(\mathcal{B}_{\nu}) = \sum_{\mathfrak{p} \text{ prime, } \operatorname{codim}_{k[\mathbf{X}]}(\mathfrak{p})=1} \operatorname{length}_{k[\mathbf{X}]_{\mathfrak{p}}}((\mathcal{B}_{\nu})_{\mathfrak{p}})[\mathfrak{p}].$$

We obtain that

$$[\det((\mathcal{K}_{\bullet})_{\nu})] = \sum_{\alpha \in \Theta} \sum_{\mathfrak{p}_{\alpha,i}} \operatorname{length}_{k[\mathbf{X}]_{\mathfrak{p}_{\alpha,i}}}((\mathcal{B}_{\nu})_{\mathfrak{p}_{\alpha,i}})[\mathfrak{p}_{\alpha,i}] + \operatorname{length}_{k[\mathbf{X}]_{(H)}}((\mathcal{B}_{\nu})_{(H)})([H]).$$

In the formula above, for each $\mathfrak{p}_{\alpha,i}$ we have

$$\operatorname{length}_{k[\mathbf{X}]_{\mathfrak{p}_{\alpha,i}}}((\mathcal{B}_{\nu})_{\mathfrak{p}_{\alpha,i}}) = \dim_{\kappa(\mathfrak{p}_{\alpha,i})}(\mathcal{B}_{\nu} \otimes_{k[\mathbf{X}]_{\mathfrak{p}_{\alpha,i}}} \kappa(\mathfrak{p}_{\alpha,i})) = \mu_{\alpha,i} \cdot \deg(\phi_{\alpha,i}),$$

where $\kappa(\mathfrak{p}_{\alpha,i}) := k[\mathbf{X}]_{\mathfrak{p}_{\alpha,i}}/\mathfrak{p}_{\alpha,i}k[\mathbf{X}]_{\mathfrak{p}_{\alpha,i}}$. Consequently we get that for each $\alpha \in \Theta$, there is a factor of G , denoted by $H_{\alpha,i}$, that corresponds to the irreducible implicit equation of the scheme theoretic image of $\phi_{\alpha,i}$, raised to a certain power $\mu_{\alpha,i} \cdot \deg(\phi_{\alpha,i})$. \square

Remark 23. It is important to remark that the set-theoretic approach doesn't tell us anything about the scheme structure of the fibre $(\pi_1)^{-1}(W)$. Namely the bijection

$$\bigcup_{\alpha} (X_{\alpha} \times ((\mathbb{P}^1)^{|\alpha|} \times \{*\} \times \dots \times \{*\})) \xrightarrow{\sim} (\pi_1)^{-1}(W)$$

is not necessarily a scheme isomorphism; for instance, $\mathcal{E}_{\alpha,i}$ need not be well-defined over all $X_{\alpha,i}$, because the multiplicities of the components of $X_{\alpha,i}$ of $\operatorname{Proj}(A/I^{\alpha})$ are not necessarily preserved by π_1^{-1} . The fiber bundle $\mathcal{E}_{\alpha,i}$ is defined over the relative open set $\Omega_{\alpha,i}$ of $X_{\alpha,i} \subset \operatorname{Proj}(A/I^{\alpha})$, but the dimension of the fibre can increase on $X_{\alpha,i} - \Omega_{\alpha,i}$.

Remark 24. Let us discuss briefly how the exponents $\mu_{\alpha,i}$ behave in the particular case where $X_{\alpha,i}$ is a complete intersection subscheme of \mathbb{P}^n . Take $\alpha \in \Theta$, defined as above, write $\alpha = \{i_0, \dots, i_k\}$, and fix $\mathfrak{p}_i \in \Lambda_{\alpha}$. As $\operatorname{codim}(I^{\alpha,i}) = |\alpha|$, we have $\dim_{\mathbb{P}^n}(X_{\alpha,i}) = n - |\alpha| = n - (k+1) =: m$, and assume $X_{\alpha,i}$ is irreducible.

Take G_0, \dots, G_k irreducible generators of $I^{\alpha,i}$ (recall $X_{\alpha,i}$ is a complete intersection), and $G_j := L_{i_j}$ for $j \in \{k+1, \dots, n\}$.

As G_0, \dots, G_k vanish over $X_{\alpha,i}$, the element $\operatorname{Res}(G_0, \dots, G_n) \in k[x_{i_{k+1}}, y_{i_{k+1}}, \dots, x_{i_n}, y_{i_n}]$ describes exactly the irreducible implicit equation, $H_{\alpha,i}$, of the scheme theoretic image, $\mathcal{H}_{\alpha,i} \subset \mathbb{P}_{i_{k+1}}^1 \times \dots \times \mathbb{P}_{i_n}^1$, of the restricted and correstricted map $\phi_{\alpha,i}$.

Assume that L_j lies in $G := (G_0, \dots, G_n)^{\mu_j}$, and that μ_j is maximum with this property, for all j . Then, from the ‘‘Lemme de divisibilit  g n ral’’ by J.-P. Jouanolou (see (Jou91, Prop. 6.2.1)), we have

$\operatorname{Res}(G_0, \dots, G_n) = H_{\alpha,i}^{\prod_j \mu_j}$. As this polynomial divides $H_{\alpha,i}^{\mu_{\alpha,i}}$, $\prod_j \mu_j$ gives a lower bound for $\mu_{\alpha,i}$.

Remark also that L_{i_j} always lies in the ideal $G := (G_0, \dots, G_n)^{\mu_j}$ for $\mu_{i_j} = 1$ by definition, when $j > k$.

5. Examples

In this section we present several examples where we illustrate the theory developed in the previous sections. These computations were done in *Maple 11*, by means of the routines implemented in the *Multires* developed by Galaad Team at INRIA, cf. (BM).

Example 25. In this example we show what happens when there is a point whose fibre has dimension 2.

```
> read"multires.mpl": with(linalg):
> f0 := u: g0 := v: f1 := u^2: g1 := v^2: f2 := v^2: g2 := w^2:
      2      2      2      2
L0 := x0 v - y0 u,   L1 := x1 v - y1 u,   L2 := x2 w - y2 v
```

```
> M012w:=det(mresultant([L0,L1,L2],[u,v]));
```

$$M012w := -x_2^2 w^4 y_1 (y_1^2 x_0^2 - x_1^2 y_0^2) y_2^2$$

As *mresultant* gives a multiple of the desired Macaulay resultant, computing the greatest common divisor over all permutations of L_0, L_1, L_2 we get that Mw should be:

$$Mw := x_2^2 w^4 (y_1^2 x_0^2 - x_1^2 y_0^2)$$

It remains to observe that x_2^2 correspond to the equation $L_2(p) = 0$, where $p = (0 : 0 : 1)$ is the point with 2-dimensional fibre, and use the fact that p has multiplicity 2.

Example 26. In this example we study the case of a two point (p_1 and p_2) base locus, where the fibre above them has dimension 2. Take $p_1 = (1 : 0 : 0)$ and $p_2 = (0 : 0 : 1)$. The computation below shows that in this case the method yields a power of the irreducible implicit equation with some extra factors. Those factors are powers of y_1 and x_2 .

Why y_1 and x_2 ? If we look at the linear equations L_0, L_1, L_2 evaluated in the two points, p_1, p_2 , we see that $L_0(p_1) = L_2(p_1) = 0$ and $L_1(p_1) = y_1$. If we do the same with p_2 we get $L_0(p_2) = L_1(p_2) = 0$ and $L_2(p_2) = x_2$.

```
> f0 := u*w; g0 := v^2; f1 := u^2; g1 := v^2; f2 := v^2; g2 := w^2;
> L0:=x0*g0-y0*f0; L1:=x1*g1-y1*f1; L2:=x2*g2-y2*f2;
```

As we did before, we compute the greatest common divisor over all the permutation of L_0, L_1, L_2 . The output Mw, Mv and Mu is:

$$Mw := x_2^2 w^8 y_1^3 (-y_2^2 x_1 y_0^2 + x_2^2 x_0^2 y_1^2)$$

$$Mv := v^8 y_1^3 x_2^4 (-y_2^2 x_1 y_0^2 + x_2^2 x_0^2 y_1^2)$$

$$Mu := y_1^2 u^8 x_2^4 (-y_2^2 x_1 y_0^2 + x_2^2 x_0^2 y_1^2)$$

The resultant can be obtained as the gcd of the three equations above.

Remark 27. In the spirit of Remark 24 take $\alpha = \{0, 1\}$, that is, $X_\alpha = V(f_0, g_0, f_1, g_1) = V(u, v)$, and consider $G_0 = u$, $G_1 = v$, and $G_2 = L_2 = y_2 v^2 - x_2 w^2$. By the multiplicativity of the resultant (see for instance (Jou91, Sec. 5.7)) we have $\text{Res}(G_0, G_1, G_2) = \text{Res}(u, v, -x_2 w^2) = -x_2$. Now, as $L_0 \in G := (G_0, G_1, G_2)$, but not in G^2 , and $L_1 \in G^2$, but not in G^3 , we have $\mu_0 = 1$ and $\mu_1 = 2$. Hence, $\text{Res}(G_0, G_1, G_2)^2 = x_2^2$ divides $\text{Res}(L_0, L_1, L_2)$, as the above computation showed. Moreover, in this case, we see that $\text{Res}(G_0, G_1, G_2)^2 = x_2^2$ coincides exactly with the extra factor.

Example 28. In this example we study the situation where the fibre along a 1-dimensional closed subscheme of \mathbb{P}^2 is \mathbb{P}^1 , and is \mathbb{P}^2 above the points $(1 : 0 : 0)$ and $(0 : 0 : 1)$.

```
> f0 := u*v; g0 := u*w; f1 := u^2+v^2; g1 := v^2; f2 := v^2; g2 := w^2;
> L0:=x0*g0-y0*f0; L1:=x1*g1-y1*f1; L2:=x2*g2-y2*f2;
```

$$Mw := w^8 x_2^2 y_1^3 y_2^3 (x_1^2 - y_1^2) (-x_0^2 y_2^2 + y_0^2 x_2^2)$$

$$Mv := v^8 x_2^4 y_1^3 (x_1^2 - y_1^2) (-x_0^2 y_2^2 + y_0^2 x_2^2)$$

$$Mu := w^8 x_2^2 y_1^2 (x_1^2 - y_1^2)$$

$$\text{Mu} := u \ y_1 \ (-x_0 \ y_2 + y_0 \ x_2)$$

This last equation corresponds to the situation on the open set $u \neq 0$. It is clear that the extra factor y_1 is appearing because of the 2 dimensional fibre above $p = (1 : 0 : 0)$, where $L_0(p) = L_2(p) = 0$, and $L_1(p) = u^2 y_1$. Similarly x_2 appears because of the 2 dimensional fibre above the point $q = (0 : 0 : 1)$, where $L_0(q) = L_1(q) = 0$, and $L_2(q) = w^2 x_2$, as was shown in the second example. The other factor that appears is $(x_1 - y_1)^2$, and it is due to the existence of a 1 dimensional closed subvariety V . In this case, we see that the dimension of the fibre over a point p is 1, for any p in a suitable relative open subset of V .

Precisely, we consider the fibre along the closed subvariety of \mathbb{P}^2 , $V(u) = \{(u : v : w) : u = 0\}$ and we project from this closed subvariety of the incidence variety to the space $\mathbb{P}^1 \times \mathbb{P}^1$ corresponding to the second two copies (because the first one would be $(0 : 0)$). If we compute the implicit equation by the method before, for the map

$$\begin{aligned} \phi|_{(u=0)} : (u=0) &\cong \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ (v : w) &\mapsto (f_1 : g_1) \times (f_2 : g_2), \end{aligned}$$

we get as output: $Mv := -v^4 (x_1 - y_1)^2$ and $Mw := x^4 (x_1 - y_1)^2 x_2^2$. Observe that the x_2^2 appearing in the second equation is still the extra factor coming from the big fibre over the point q in the closed set $V(u)$. We see here that the dimension of the fibre is not constant on $V(u)$, but on a relative open set where it defines a fibre bundle \mathcal{E} .

6. Applications to the computation of sparse discriminants

The computation of sparse discriminants is equivalent to the implicitization problem for a parametric variety, to which we can apply the techniques developed in the previous sections. In the situation described in (CD07), a rational map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ given by homogeneous rational functions of total degree zero is associated to an integer matrix B of full rank. This is done in such a way that the corresponding implicit equation is a dehomogenization of a sparse discriminant of generic polynomials with exponents in a Gale dual of B .

Suppose for instance that we take the matrix B below:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, as the columns of B generate the affine relations of the lattice points $\{0, 1, 2, 3, 4\}$. The closed image of the parametrization f is a dehomogenization of the classical discriminant of a generic univariate polynomial of degree 4. Explicitly, from the matrix we get the linear forms $l_1(u, v, w) = u$, $l_2(u, v, w) = -2u + v$, $l_3(u, v, w) = u - 2v + w$, $l_4(u, v, w) = v - 2w$, $l_5(u, v, w) = w$ (whose coefficients are read in the rows of B), and the polynomials $f_0 = l_1 \cdot l_3$, $g_0 = l_2^2$, $f_1 = l_2 \cdot l_4$, $g_1 = l_3^2$, $f_2 = l_3 \cdot l_5$, $g_2 = l_4^2$ (the exponents of the linear forms are read from the columns of B). This construction gives rise to the following rational map:

$$\begin{aligned} f : \mathbb{C}^3 &\dashrightarrow \mathbb{C}^3 \\ (u, v, w) &\mapsto \left(\frac{u(u - 2v + w)}{(-2u + v)^2}, \frac{(-2u + v)(v - 2w)}{(u - 2v + w)^2}, \frac{(u - 2v + w)w}{(v - 2w)^2} \right). \end{aligned}$$

First, we see that we can get a map from $\mathbb{P}_{\mathbb{C}}^2$ because of the homogeneity of the polynomials. Also, taking common denominator, we can have a map to $\mathbb{P}_{\mathbb{C}}^3$, this is:

$$f : \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^3$$

$$(u : v : w) \mapsto (f_0 : f_1 : f_2 : f_3).$$

where $f_0 = (-2u + v)^2(u - 2v + w)^2(v - 2w)^2$ is the common denominator, $f_1 = u(u - 2v + w)^3(v - 2w)^2$, $f_2 = (-2u + v)^3(v - 2w)^3$ and $f_3 = (u - 2v + w)w(-2u + v)^2(u - 2v + w)^2$.

The problem with this way of projectivizing is that, in general, we cannot implement the theory developed by L. Busé, M. Chardin, and J-P. Jouanolou, because typically the base locus has awful properties, as a consequence of taking common denominator.

As a possible way out, we propose in this work to consider the morphism of projective schemes given by:

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

$$(u : v : w) \mapsto (f_0 : g_0) \times (f_1 : g_1) \times (f_2 : g_2).$$

where $f_0 = u(u - 2v + w)$, $g_0 = (-2u + v)^2$, $f_1 = (-2u + v)(v - 2w)$, $g_1 = (u - 2v + w)^2$, $f_2 = (u - 2v + w)w$, $g_2 = (v - 2w)^2$. For this particular example, we get that there are only two base points giving rise to an extra factor, namely $p = (1 : 2 : 3)$ and $q = (3 : 2 : 1)$. Is easy to see that those points give rise to two linear factors in the resultant.

First, we observe that this situation is better, because we are not adding common zeroes. Moreover, if a point $(u : v : w)$ is a base point here, it also is in the two settings above.

Remember also that in the $n = 2$ case, the condition required on the Koszul complex associated to this map for being acyclic is that the variety X , defined as the common zeroes of all the 6 polynomials, be empty. In general, the conditions we should check are the ones imposed by the Avramov's theorem, as was shown in Theorem 13.

Note also that if we want to state this situation in the language of approximation complexes, we need only to replace \mathcal{K}_{\bullet} by \mathcal{Z}_{\bullet} , because we can assume that $\{f_i, g_i\}$ are regular sequences, due to the fact that $\gcd(f_i, g_i) = 1$.

Remark 29. For a matrix like the B above, it is clear that the closed subvariety X is always empty, due to the fact that all maximal minors of B are not zero, and the polynomials g_i 's involve independent conditions. Then, the only common solution to $l_2^2 = l_3^2 = l_4^2 = 0$ is $(u, v, w) = (0, 0, 0)$, and so $X = \emptyset$ in \mathbb{P}^2 . In this case, it is still better (from an algorithmic approach) to compute the discriminant of a generic polynomial of degree 4 in a single variable and then dehomogenize, because, in our setting, the number of variables is bigger than 1. But when the number of monomials of a sparse polynomial in many variables is not big, this Gale dual approach for the computation of sparse discriminants provides a good alternative.

We will give next an example where we show a more complicated case.

Example 30. Let C be the matrix given by

$$C = \begin{pmatrix} 1 & -7 & -6 \\ -1 & 4 & 3 \\ 1 & 0 & 4 \\ 0 & 1 & -1 \\ -1 & 2 & 0 \end{pmatrix}.$$

As before, denoting by b_i the i -th row of C , we get the linear forms $l_i(u, v, w) = \langle b_i, (u, v, w) \rangle$, associated to the row vectors b_i of B , where \langle, \rangle stands for the inner product in \mathbb{C}^3 . Then we define the homogeneous polynomials $f_0 = l_1 \cdot l_3$, $g_0 = l_2 \cdot l_5$, $f_1 = l_2^4 \cdot l_4 \cdot l_5^2$, $g_1 = l_1^7$, $f_2 = l_2^3 \cdot l_3^4$, $g_2 = l_1^6 \cdot l_4$. And we obtain the following rational map:

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

$$(u : v : w) \mapsto (f_0 : g_0) \times (f_1 : g_1) \times (f_2 : g_2).$$

It is easy to see that in this case the variety X is not empty, for instance the point $p = (1 : 1 : -1)$, defined by $l_1 = l_2 = 0$ belongs to X .

As was shown by M. A. Cueto and A. Dickenstein in (CD07, Lemma 3.1 and Thm. 3.4), we can interpret the discriminant computed from the matrix C in terms of the dehomogenized discriminant associated to any matrix of the form $C \cdot M$, where M is a square invertible matrix with integer coefficients. That is, we are allowed to do operations on the columns of the matrix C , and still be able to compute the desired discriminant in terms of the matrix obtained from C . In (CD07) they give an explicit formula for this passage.

In this particular case, we can multiply C from the right by a determinant 1 matrix M , obtaining

$$C \cdot M = \begin{pmatrix} 1 & -7 & -6 \\ -1 & 4 & 3 \\ 1 & 0 & 4 \\ 0 & 1 & -1 \\ -1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 12 & -1 \\ 0 & 6 & -1 \\ 0 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -3 & 0 \\ 1 & -8 & 3 \\ 0 & 11 & -2 \\ -1 & 0 & -1 \end{pmatrix}.$$

Similar to what we have done before, we can see that the closed subvariety X associated to the rational map that we obtain from the matrix $C \cdot M$ is empty. Observe that $\#V(I_2)$ is finite due to the fact that $l_2 = l_4 = 0$ or $l_3 = l_4 = 0$ or $l_3 = l_5 = 0$ should hold. Moreover it is easy to verify that all maximal minors are nonzero, and this condition implies that any of the previous conditions define a codimension 2 variety, this is, a finite one. A similar procedure works for seeing that $\text{codim}_A(I_3) \geq 2$. Finally the first part of Theorem 13 implies that the Koszul complex \mathcal{K}_\bullet is acyclic and so we can compute the Macaulay resultant as its determinant.

Moreover, this property over the minors implies that $\text{codim}_A(I^{(i_0)}) = 2 > k+1 = 1$ and that $\text{codim}_A(I^{(i_0)} + I^{(i_1)}) = 3 > k+1 = 2$. So, the second part of Theorem 13 tells us that the determinant of the Koszul complex \mathcal{K}_\bullet in degree greater than $(2+8+3)-3 = 10$ determines exactly the implicit equation of the scheme theoretic image of ϕ . Observe that, as was shown in (CD07, Thm. 2.5), for this map, we have that $\deg(\phi) = 1$.

We remark that the process implemented for triangulating the matrix C via M is not algorithmic for the moment.

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